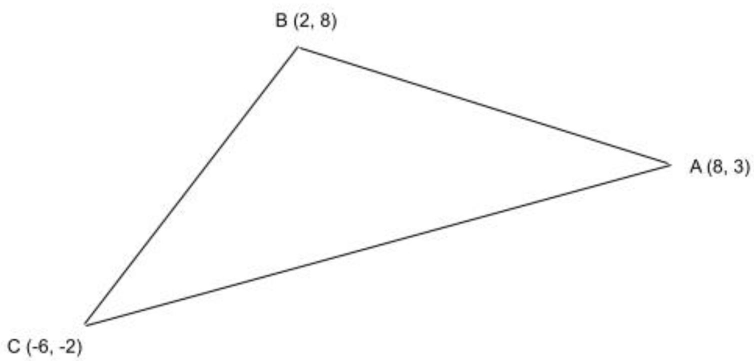
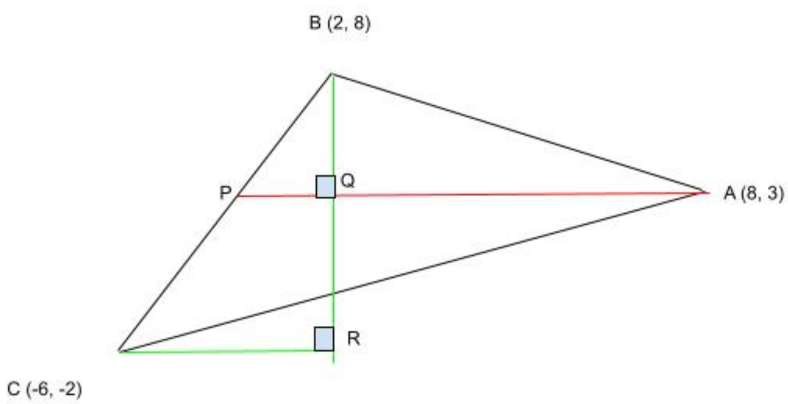
Understanding the Surveyor's Theorem by Janet Johnson

We begin with the most basic formula for finding the area of a triangle $A = \frac{1}{2}bh$. This formula however, is not always the most convenient approach to finding area. The following example illustrates this point:



No matter which side of the triangle is used as the base, finding the height requires tedious algebraic calculations.

Suppose we find the point of the triangle that is on the same vertical or horizontal line as one of the vertices. For instance, point P would be horizontal to point A and located on line BC. The following calculations will give the coordinates of P.



Since triangle BPQ is similar to triangle BCR, the following segments are proportional:

$$\frac{PQ}{CR} = \frac{BO}{BR} \quad so \quad \frac{PQ}{CR} = \frac{5}{10}$$

Hence, $CR = 2PQ$. Since $CR = 8$, then $PQ = 4$. Now, we know the coordinates of P are (-2, 3).

If we use AP as the base of both triangles ABP and APC, the area of triangle ABC is the sum of the two and the calculations are elementary:

$Area_{ABP} = \frac{1}{2}(10)(5) = 25$ and $Area_{APC} = \frac{1}{2}(10)(5) = 25$ so $Area_{ABC} = 50$

Note: $Area_{ABC} = Area_{ABP} + Area_{APC} = \frac{1}{2}(AP \cdot BQ + AP \cdot QR) = \frac{1}{2}AP(BQ + QR) = \frac{1}{2}AP \cdot BR$

$$= \frac{1}{2}hor(A,P) \cdot ver(B,C)$$

$$= \frac{1}{2}hor(A, line\ BC) \cdot ver(B,C)$$

Where $hor(A\ P)$ is the horizontal distance from A to P and $ver(B,C)$ is the vertical distance from B to C.

Area Theorem: *Given points A, B, and C,*

(signed) $Area_{ABC} = \frac{1}{2}hor(A, line\ BC) \cdot ver(B,C)$

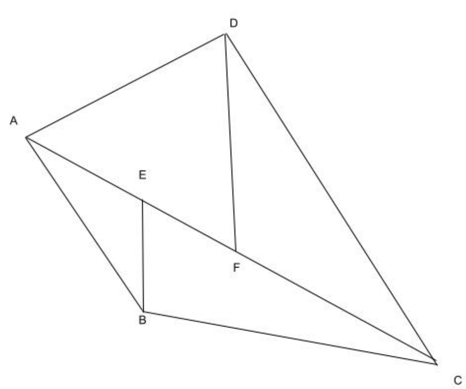
The following Lemma can be used to find the area of a quadrilateral:

Lemma: *Given collinear points D, E, F, and B, we have*

$hor(D,B) \cdot ver(E,F) = hor(E,F) \cdot ver(D,B)$

We omit the proof which is elementary and follows from considering similar triangles.

Let's use the following example to show how this Lemma can be useful in finding the area of a quadrilateral:



$$Area_{ABCD} = Area_{ABD} + Area_{BCD} = \frac{1}{2}ver(B,E)hor(A,C) + \frac{1}{2}ver(F,D)hor(A,C)$$

Factoring out common terms gives:

$=\frac{1}{2}hor(A,C)[ver(B,E) + ver(F,D)]$

$$= \frac{1}{2}hor(A,C)[ver(B,D) + ver(F,E)]$$

Using the Lemma and redistributing gives:

$$= \frac{1}{2}[hor(A,C)\cdot ver(B,D) + hor(A,C)\cdot ver(F,E)]$$

$$= \frac{1}{2}[hor(A,C)\cdot ver(B,D) - ver(A,C)\cdot hor(E,F)]$$

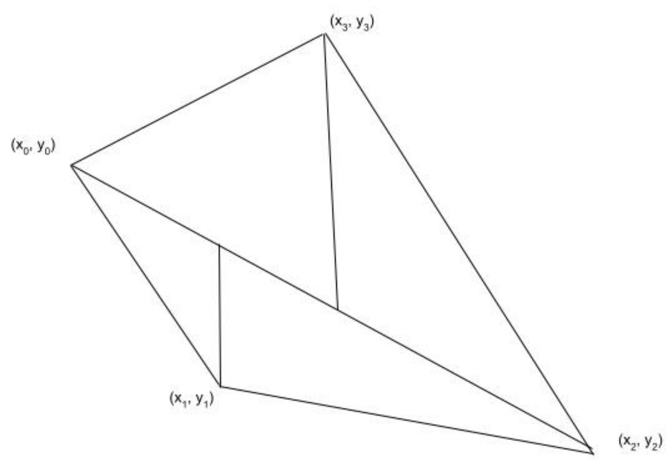
$$= \frac{1}{2}[hor(A,C)\cdot ver(B,D) - hor(B,D)\cdot ver(A,C)$$

Where the last equality holds true because the horizontal distance from E to F is the same as the horizontal distance from B to D.

We proved: $\mathrm{Area}_{\mathrm{ABCD}} = \frac{1}{2}[hor(A,C)\cdot ver(B,D) - hor(B,D)\cdot hor(A,C)]$

Note that hit can be written as $Area_{ABCD} = \frac{1}{2}(\overrightarrow{AC}_x\overrightarrow{BD}_y - \overrightarrow{BD}_x\overrightarrow{AC}_y)$ which can also be written using the cross-product in R^3. $Area_{ABCD} = \frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{BD})$

Let's look at our example above; this time, however, replacing ABCD with coordinates.



We know $Area_{ABCD} = \frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{BD}\right) = \frac{1}{2}(\overrightarrow{AC}_x\overrightarrow{BD}_y - \overrightarrow{BD}_x\overrightarrow{AC}_y)$

Using the coordinates above, this becomes: $\frac{1}{2}[(x_2 - x_0)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_0)]$

Distribution leads to: $\frac{1}{2}(x_2y_3 - x_2y_1 - x_0y_3 + x_0y_1 - x_3y_2 + x_3y_0 + x_1y_2 - x_1y_0)$

Rearranging terms gives: $\frac{1}{2}[(x_0y_1 - x_1y_0) + (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_0 - x_0y_3)]$

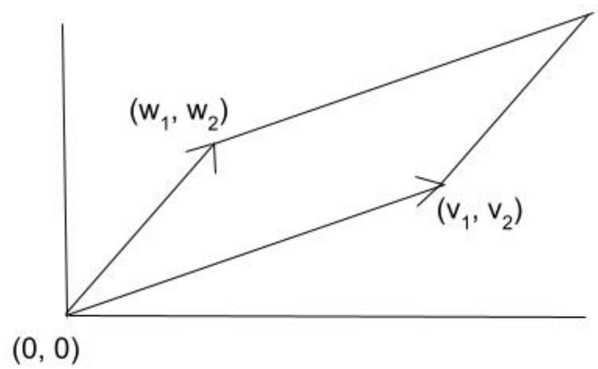
Finally, rewriting the groups as determinants, we find:

$$Area = \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_0 \\ y_3 & y_0 \end{vmatrix}$$

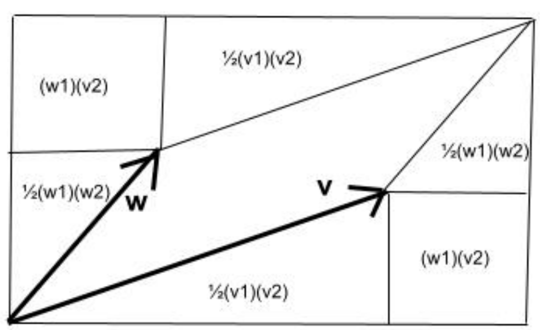
This interpretation of area is known as The Surveyor's Area Formula. It can be used to find the area of any simple polygon that has no intersection or holes with vertices $(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1})$ listed counterclockwise around the polygon. The proof of the formula is based on the Lemma below:

The absolute value of $\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$ is the area of the polygon with vectors V= (v_1, v_2) and W= (w_1, w_2)

To prove this let's view the following parallelogram as defined by vectors V and W.



By enclosing the parallelogram in a rectangle and defining the enclosed segments using the coordinates of the vectors, we have:



Since the area of the rectangle is $(v_1 + w_1)(v_2 + w_2)$, then the area of the parallelogram is

$$(v_1 + w_1)(v_2 + w_2) - \left(\left(\frac{1}{2}v_1v_2\right) + (w_1v_2) + \left(\frac{1}{2}w_1w_2\right) + \left(\frac{1}{2}v_1v_2\right) + (w_1v_2) + (\frac{1}{2}w_1w_2)\right)$$

Combining like terms gives:

$$(v_1 + w_1)(v_2 + w_2) - \big((v_1v_2) + (2w_1v_2) + (w_1w_2)\big)$$

Distributing the first two factors gives:

$$v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2 - \big((v_1v_2) + (2w_1v_2) + (w_1w_2)\big)$$

Distributing the negative gives:

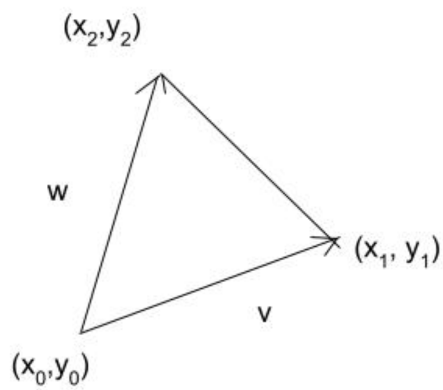
$v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2 - v_1v_2 - 2w_1v_2 - w_1w_2$

Finally, eliminating terms that cancel yields:

$v_1w_2 - v_2w_1$

Hence, the area of the parallelogram is $\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$

Now, let's expand the Surveyor's formula to the area of a triangle. From the previous example, we know the area of a parallelogram is $|v_1w_2 - v_2w_1|$. In terms of the figure below, this expression would be $\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}$



For a triangle, the area would be $\frac{1}{2}\begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}$

Written as a 3X3 determinant, the following are equivalent:

$$A = \frac{1}{2}\begin{vmatrix} x_1 - x_0 & x_0 - x_2 \\ y_1 - y_0 & y_0 - y_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ x_0 & x_1 - x_0 & x_2 - x_0 \\ y_0 & y_1 - y_0 & y_2 - y_0 \end{vmatrix}$$

Now, by replacing the second column with the sum of the first and second columns, and replacing the third column with the sum of the first and third columns, we have

$$A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix}$$

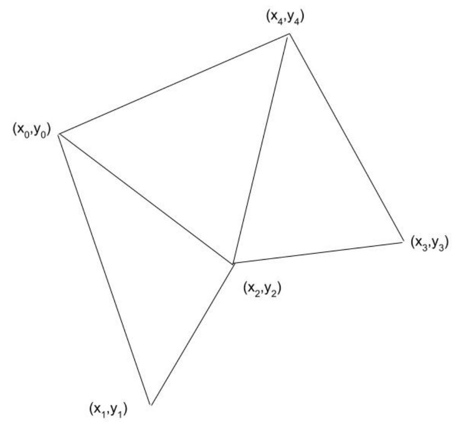
The expansion of this determinant gives:

$$A = \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix} + \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}\right)$$

Rearranging the first and third determinants, and switching the columns in the second matrix allows us to write

$$A = \frac{1}{2}\left(\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix}\right)$$

From this, we obtain the Surveyor's Formula. Knowing that a simple polygon can be triangulated, this formula can be used to find the area of a polygon with side lengths greater than three.
We will use the following polygon to illustrate:



The area of the polygon is the sum of the areas of the triangles. So the area of the polygon is the sum of the determinant as follows:

$$A = \frac{1}{2}\left(\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix} + \begin{vmatrix} x_2 & x_4 \\ y_2 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_0 \\ y_4 & y_0 \end{vmatrix} + \begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_2 \\ y_4 & y_2 \end{vmatrix}\right)$$

Notice that several determinants will cancel. For example $\begin{vmatrix} x_2 & x_0 \\ y_2 & y_0 \end{vmatrix}$ and $\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}$ will cancel each other.

Also, $\begin{vmatrix} x_2 & x_4 \\ y_2 & y_4 \end{vmatrix}$ and $\begin{vmatrix} x_4 & x_2 \\ y_4 & y_2 \end{vmatrix}$ will cancel one another. The expression we are left with is

$$A = \frac{1}{2}\left(\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_0 \\ y_4 & y_0 \end{vmatrix}\right)$$

So for any polygon of side n, the area is

$$A = \frac{1}{2}\left\{\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-2} & x_{n-1} \\ y_{n-2} & y_{n-1} \end{vmatrix} + \begin{vmatrix} x_{n-1} & x_0 \\ y_{n-1} & y_0 \end{vmatrix}\right\}$$

References

1. S.Anderson and O.Dyer. Finding Polygonal Areas with the Corset Theorem, *College Math J.* (2017) 171-178, http://jstor.org/stable/10.4169/college.math.j.48.3.171

2. B. Braden. The surveyor's area formula, *College Math*. J. (1986) 326-337, http://jstor.org/stable/2686282

3. S. Golomb . Proof without Words: A 2X2 Determinant Is the Area of a Parallelogram, mathematics magazine (1985) 107, https://www.maa.org/programs/faculty-and-departments/classroom-capsules-and-notes/proof-without-words-a-2-x-2-determinant-is-the-area-of-a-parallelogram